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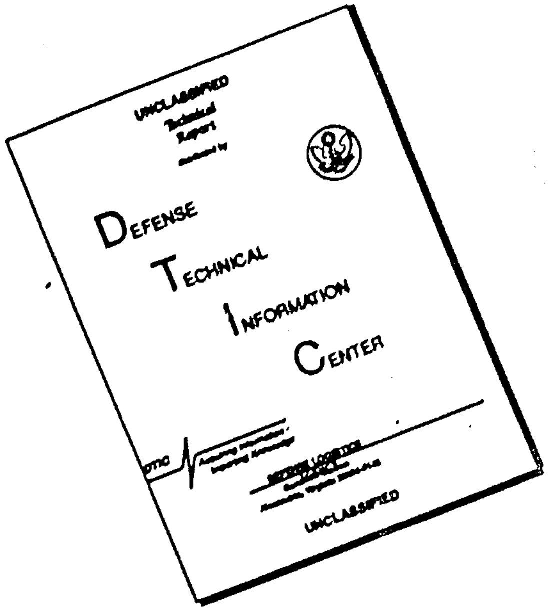

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THE KNAPSACK PROBLEM: SOME RELATIONS FOR AN IMPROVED ALGORITHM

by

Fred Glover

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Graduate School of Industrial Administration

Carnegie Institute of Technology

Pittsburgh, Pennsylvania 15213

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ABSTRACT

The knapsack problem has traditionally been solved by dynamic programming, though a more recent enumerative algorithm by P. C. Gilmore and R. E. Gomory has proved somewhat more efficient. In this paper relations are developed which substantially reduce the number of solutions which must be examined by the Gilmore-Gomory method, or by any algorithm for the knapsack problem which uses an enumerative base. Our results enable certain problems for which the Gilmore-Gomory method reduces almost to complete enumeration to be solved after examining only a handful of alternatives. Computer studies to provide detailed comparisons of methods which do and do not employ these results have not yet been undertaken.

Form of the Problem.

The knapsack problem may be written

(1) Maximize cx

subject to $ax \leq b$, $x \geq 0$ and x integer,

where $c = (c_1 \ c_2 \ \ldots \ c_n)$ and $a = (a_1 \ a_2 \ \ldots \ a_n)$ are $1 \times n$ row vectors of positive constants, b is a positive scalar, and $x = (x_1 \ x_2 \ \ldots \ x_n)^T$ is an $n \times 1$ column vector of nonnegative integer variables. Frequently (1) is accompanied by an additional set of restrictions of the form $x_i \leq \alpha_i$ for $i = 1, \ldots, n$, in which case we will refer to it as the bounded variable knapsack problem.

Practical application for the knapsack problem of some significance has been found by P. C. Gilmore and R. E. Gomory. In references 3 and 4 these authors show how the knapsack problem may be used to enable certain linear programming problems involving an enormous number of variables to be usefully handled with linear programming methods. With the introduction of a special enumerative algorithm in [4], Gilmore and Gomory found it possible to obtain solutions for the knapsack problem about five times more rapidly than with dynamic programming, and thereby were able to use their method as an adjunct to the linear programming algorithm to solve problems from the paper industry within reasonable time limits on the computer. Nonetheless, computation devoted to solving the knapsack problems was a sizable fraction of total computation, so that improved algorithms for problem (1) would seem to be assured of immediate application.

In what follows we will first outline the enumerative algorithm for (1) proposed by Gilmore and Gomory, and then by means of discussion and theorems develop relations designed to exclude a substantial number of solutions from consideration when applying any such enumerative method.

Gilmore and Gomory's Method.

It is customary in dealing with problem (1) to index each variable x_i in terms of its contribution c_i to the objective function value, divided by the amount a_i by which this variable occupies a portion of the available "space" in the constraint $ax \leq b$. This practice appears to have originated with Dantzig [2], and will be adhered to throughout the remainder of this paper; i.e., we assume $c_1/a_1 \geq c_2/a_2 \geq \dots \geq c_n/a_n$.

The algorithm of Gilmore and Gomory uses this indexing as the basis for a lexicographic ordering of the solutions to be enumerated, and as a means for testing when certain solutions may be bypassed in the enumeration.[1] It may be observed that the lexicographically largest solution $\bar{x}$ which satisfies $ax \leq b$ is given by

$$\bar{x}_1 = [b/a_1], \text{ and}$$
$$\bar{x}_k = [(b - \sum_{i=1}^{k-1} a_i \bar{x}_i)/a_k] \text{ for } k = 2, \dots, n,$$

since clearly no value of x_1 can exceed $\bar{x}_1$, and given $x_1 = \bar{x}_1$, no value of x_2 can exceed $\bar{x}_2$, etc.[2] Also, by defining $s = \max(i \mid \bar{x}_i \neq 0)$, it may be seen that for any feasible solution x to (1), the lexicographically largest solution $\hat{x}$ which is lexicographically smaller than $\bar{x}$ is given by

$$\hat{x}_k = \begin{cases} \bar{x}_k & \text{for } k < s \\ \bar{x}_k - 1 & \text{for } k = s \\ [(b - \sum_{i=1}^{k-1} a_i \hat{x}_i)/a_k] & \text{for } k = s+1, \dots, n. \end{cases}$$

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1. The Gilmore-Gomory method is coupled with a technique which enables a number of knapsack problems having the same a and c vectors to be solved simultaneously. The basis for this technique may be seen by noting that the set of feasible solutions to (1) for $b = b_0$ is a subset of the set of feasible solutions to (1) for $b \geq b_0$.

2. We use the symbol $[z]$ to denote the greatest integer less than or equal to z, and $\langle z \rangle$ to denote the least integer greater than or equal to z.

Except for the test which rules certain solutions out of consideration, the Gilmore-Gomory method begins with the lexicographically largest $\bar{x}$, and then redefines $\bar{x}$ to equal $\hat{x}$, repeating this process until there are no solutions left to be generated. To avoid enumeration of all possibilities, the largest value M of cx obtained at any point in the process is used to rule out examination of some of those solutions $\bar{x}$ for which $c\bar{x} \leq M$. This is accomplished as follows.

For any feasible solution x', consider problem (1) subject to the additional restriction that $x_i = x_i'$ for $i = 1, \ldots, s$, where s is chosen so that $x_i' = 0$ for $i > s$. Then (1) is replaced by

$$(2) \quad \text{Maximize} \quad \sum_{i>s} c_i x_i + cx'$$

$$\text{subject to} \quad \sum_{i>s} a_i x_i \leq b - ax'.$$

If $s = n$, the objective function value for this problem is trivially equal to cx'. Otherwise, an upper bound on the value of the objective function for (2) is given by relaxing the integer requirements on the variables x_i for $i > s$. Due to the ordering of the variables the solution to the problem in this case is clearly to take the value of x_{s+1} as large as possible, or $x_{s+1} = (b - ax')/a_{s+1}$. This yields the objective function value $cx' + c_{s+1}(b - ax')/a_{s+1}$, which we will denote by $\bar{M}$. Thus, given any solution x', a necessary condition that there exist an integer solution x^* such that $cx^* > M$ and $x_i^* = x_i'$ for $i \leq s$ is just $\bar{M} > M$. Gilmore and Gomory use this fact in their algorithm by dividing the step which defines $\hat{x}$ into two parts. In the first part they define the solution x' so that

$$x_k' = \begin{cases} \bar{x}_k & \text{if } k < s \\ \bar{x}_k - 1 & \text{if } k = s \\ 0 & \text{if } k > s. \end{cases}$$

where $s = \max(i \mid \bar{x}_i \neq 0)$. If $\bar{M} > M$, the definition of $\hat{x}$ is completed as earlier, and the enumeration process continues uninterrupted. But if $\bar{M} \leq M$, all solutions which agree with x' in the first s components (including $\hat{x}$) may clearly be ignored. Thus the next step of the enumeration would be to obtain the new $\bar{x}$ which is the lexicographically largest solution lexicographically smaller than x'. But in fact the new $\bar{x}$ cannot agree with x' in its first $s - 1$ components (s still defined relative to the previous $\bar{x}$), since then $\bar{x}$ must be smaller than x' in the $s\underline{th}$ component, and the test $\bar{M} > M$ would clearly be failed again. ($\bar{M}$ cannot increase until x_i' is increased for at least one i such that $i \leq s$.) Thus, when $\bar{M} < M$, $\bar{x}_s$ is immediately set equal to 0, and the vector $\hat{x}$ is defined relative to this latter $\bar{x}$.

In conjunction with their algorithm, Gilmore and Gomory use a preliminary trimming technique which throws away a number of the problem variables before enumeration begins. Whenever $c_i = c_j$, one of the two variables x_i or x_j may be discarded, depending on the relative sizes of a_i and a_j. Thus for $c_i = c_j$ and $a_i \leq a_j$, the variable x_j may be dropped since it may always be replaced by x_i in any optimal solution. While Gilmore and Gomory note that such an equality of "prices" would not normally be expected, they found it to occur with sufficient frequency in the problems they examined to cause the average size of these problems to be reduced from 30 to 18.2 variables.

Relations for an Improved Algorithm.

In this section we will consider the more general form of problem (1) in which $x_i \leq \alpha_i$ for $i = 1, \ldots, n$, where the α_i are assumed to be integer. When $\alpha_i \geq [b/a_i]$, will we continue to refer to x_i as "unbounded." It is evident that the lexicographically largest feasible solution $\bar{x}$ to (1) for the more general formulation is given by

$$\bar{x}_1 = \min(\alpha_1, [b/a_1]), \text{ and}$$
$$\bar{x}_k = \min(\alpha_k, [(b - \sum_{i=1}^{k-1} a_i\bar{x}_i)/a_k]), \text{ for } k = 2, \ldots, n.$$

Similarly, for any feasible solution $\bar{x}$, the lexicographically largest feasible solution $\hat{x}$ which is lexicographically smaller than $\bar{x}$ is given by

$$\hat{x}_k = \begin{cases} \bar{x}_k \text{ for } k < s \\ \bar{x}_k - 1 \text{ for } k = s \\ \min(\alpha_k, [(b - \sum_{i=1}^{k-1} a_i\hat{x}_i)/a_k]) \text{ for } k > s, \end{cases}$$

where $s = \max(i \mid \bar{x}_i \neq 0)$, as before.

We will adhere to the convention that $\hat{x}$ will always be defined as above relative to any given $\bar{x}$. We will also follow the convention, unless specified otherwise, that x' will be defined as in the preceding section to be the same as $\bar{x}$ except in the sth component, in which case $x_s' = \bar{x}_s - 1$. As already noted, any solution lexicographically smaller than $\bar{x}$ must agree with x' in the first s components.

The proofs of the theorems to follow are given in the appendix.

Theorem 1. If there exist nonnegative integers h_i such that (i) $h_k > 0$, (ii) $\sum_{i\neq k} h_i c_i \geq h_k c_k$, (iii) $\sum_{i\neq k} h_i a_i \leq h_k a_k$, and (iv) $[(b - a_k h_k)/a_i] + h_i \leq \alpha_i$ for all i such that $h_i \neq 0$ and $i \neq k$, then there exists an optimal solution x^* to (1) in which $x_k^* < h_k$. Moreover, if $\sum_{i\neq k} h_i c_i > h_k c_k$, then $x_k^* < h_k$ in every optimal solution x^* to (1).

When $h_k = 1$, Theorem 1 gives a somewhat stronger criterion for ruling variables out of consideration than provided by the fortuitous condition $c_i = c_j$. One of the simpler ways to exploit the theorem when the x_i are unbounded is to drop x_j whenever an i exists such that

$\langle c_j/c_i \rangle a_i \le a_j$. Substantial reduction of the space of solutions to be enumerated may also be effected by using the theorem to provide more restrictive upper bounds for some of the problem variables.

Example.

Maximize $\quad 24x_1 + 5x_2 + 14x_3 + 6x_4 + 7x_5 + 10x_6 + 8x_7$

subject to $\quad 8x_1 + 2x_2 + 7x_3 + 3x_4 + 4x_5 + 6x_6 + 5x_7 \le 79$.

For this problem in unbounded variables, x_2 may be used to rule out all variables except x_1 and x_4, using the relation $\langle c_j/c_i \rangle a_i \le a_j$. In addition, $x_2 \le 3$ and $x_4 \le 1$ are implied by the coefficients of x_1 and x_2, respectively.

We observe that any theorem concerning problem (1) applies also to problem (2) of the preceding section, with i restricted to $i > s$, b replaced by $b - ax'$, etc. Thus when the u_i are too small to permit extensive use of Theorem 1 before the enumeration begins, it may be noted that more fruitful results may be available when $[(b - ax')/a_i]$ becomes closer to u_i. In this case, certain variables may be "temporarily" ruled out of consideration.[1]

The next three theorems present additional ways of uncovering restrictions on the x_i when a feasible solution is obtained.

Theorem 2. For any feasible solution $\bar{x}$, and for any q such that $a\bar{x} + a_q > b$, let $P_q = \{ i \mid i \ne q,\ \langle c_q/c_i \rangle \le u_i - \bar{x}_i + 1,\ \langle c_q/c_i \rangle a_i \le a_q \}$. Then there exists an optimal solution x^* such that (i) $x_q^* \le \bar{x}_q$, or (ii) $x_i^* < \bar{x}_i$ for at least one i such that $i \notin P_q$. If in the definition of P_q, $[c_q/c_i] + 1$ replaces $\langle c_q/c_i \rangle$, then (i) or (ii) must hold for every optimal solution x^*.

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1. A simple way of keeping track of such temporary restrictions is given by the author in [5], where more flexible enumeration procedures than the lexicographically decreasing solution sequence are also presented. Similar procedures are also employed by Egon Balas in the prior article [1].

One of the uses of Theorem 2 is as follows. Suppose a feasible solution $\bar{x}$ is obtained at any point in the enumeration process for which $a\bar{x} + a_q > b$ and $\bar{x}_i = 0$ for $i \notin P_q$. Then it immediately follows that $x_q^* = 0$ in some optimal solution x^*, and hence the variable x_q may be dropped from the problem. Similarly, when $\alpha_q > \bar{x}_q > 0$, and the same conditions as above otherwise obtain, $\bar{x}_q$ may replace α_q to provide a more restrictive upper bound on x_q.

Example.

Maximize $18x_1 + 17x_2 + 19x_3 + 26x_4 + 16x_5 + 12x_6 + 17x_7 + 14x_8$

subject to $3x_1 + 4x_2 + 5x_3 + 7x_4 + 5x_5 + 4x_6 + 6x_7 + 5x_8 \leq 63$

and $\alpha_i = 5$ for all i.

For this example, a feasible solution is given by $\bar{x}_i = 5$ for $i \leq 3$, and $\bar{x}_i = 0$ for $i > 3$. Hence, applying Theorem 2, it follows that x_5, x_7, and x_8 may be eliminated from consideration in solving for the optimum.

Theorem 3. For any feasible solution $\bar{x}$ in which there exists a component $\bar{x}_p > 0$, let $Q_p = \left\{ i \mid i \neq p,\ \langle c_i/c_p \rangle \leq \alpha_p - \bar{x}_p + 1,\ \langle c_i/c_p \rangle a_p \leq a_i \right\}$. Then there exists an optimal solution x^* in which (i) $x_p^* \geq \bar{x}_p$ or (ii) $x_i^* > \bar{x}_i$ for at least one i such that $i \notin Q_p$. If in the definition of Q_p, $\langle c_i/c_p \rangle$ is replaced by $[c_i/c_q] + 1$, then (i) or (ii) holds for every optimal solution x^* to (1).

Theorem 3 is very nearly a reverse image of Theorem 2. One use for this theorem occurs when a feasible solution $\bar{x}$ is obtained for which $\bar{x}_p > 0$ and $\bar{x}_i = \alpha_i$ for $i \notin Q_p$ (except possibly for $i = p$). The problem may then be simplified by replacing x_p with the new nonnegative integer variable $z_p = x_p - \bar{x}_p$, which has the upper bound $\alpha_p - \bar{x}_p$. (If $\alpha_p - \bar{x}_p = 0$, then z_p may of course be dropped.) A second use is given by the following corollary.

Corollary to Theorem 3. Let $\bar{x}$ be a feasible solution for (1), and let $s = \max(i \mid \bar{x}_i \neq 0)$. Then $\bar{x}$ is optimal if (i) $a_i > b - a\bar{x}$ for $i > s$, (ii) $\bar{x}_i = a_i$ for $i \leq s$, and (iii) for each i and p such that $p \leq s$, $i > s$, $c_p \geq c_i$ and $a_p \leq a_i$.

Example.

Maximize	$11x_1$	$+ 13x_2$	$+ 9x_3$	$+ 10x_4$	$+ 8x_5$	$+ 9x_6$	$+ 6x_7$		
subject to	$2x_1$	$+ 4x_2$	$+ 3x_3$	$+ 4x_4$	$+ 4x_5$	$+ 5x_6$	$+ 4x_7$	$\leq$	82
$a_i =$	10	6	5	5	7	3	5		

Applying the corollary to this problem reveals that the solution $\bar{x}$ obtained by setting $\bar{x}_i = a_i$ for $i \leq 4$ and $\bar{x}_i = 0$ for $i > 4$ is optimal. It is to be noted that the corollary provides a sufficient condition for the lexicographically largest feasible solution to (1) to be optimal. When $\hat{x}$ replaces $\bar{x}$ in the enumeration process, by extension the corollary may also be used to give a sufficient condition that $\hat{x}$ is optimal from among those which agree with $\hat{x}$ in the first s components, providing a means for shortcutting the enumeration when the proper conditions obtain.

A theorem of a somewhat different nature, but which may also be used to identify an optimal solution, or prescribe bounds for certain of the x_i, is as follows.

Theorem 4. Let $\bar{x}$ and s be given as in the preceding corollary, and let q be any subscript greater than s. Let $G = b - a\bar{x}$, and let h be a positive integer such that $a_q > G/h$. Then if $c_q/(a_q - G/h) \leq c_s/a_s$, there exists an optimal solution x^* such that (i) $x_q^* < h$ or (ii) $x_i^* > \bar{x}_i$ for at least one i such that $i \leq s$. If $c_q/(a_q - G/h) < c_s/a_s$, then (i) or (ii) holds for all optimal solutions x^* to (1).

<u>Example.</u>

Maximize $\quad 80x_1 + 106x_2 + 57x_3 + 105x_4 + 70x_5 + 40x_6 + 97x_7 + 85x_8$

subject to $\quad 10x_1 + 14x_2 + 8x_3 + 15x_4 + 12x_5 + 7x_6 + 18x_7 + 16x_8 \leq 380$

and $\alpha_i = 8$ for all i.

The feasible solution for this problem obtained by setting $\bar{x}_i = \alpha_i = 8$ for $i \leq 4$, and $x_i = 0$ for $i > 4$, yields a value for G ($= b - a\bar{x}$) of 4. To find the most restrictive bounds on the x_q for $q > 4$ which can be derived from thissolution, we wish to minimize h for each such variable, subject to the restrictions of the theorem. Thus for each $q > 4$ it may be readily verified that the best value of h is given by $h = \max([G/a_q] + 1, \langle c_s G/(c_s a_q - c_q a_s)\rangle)$. This yields values of h for x_4 through x_8 of 2, 4, 1, 2, respectively. Hence, upper bounds for these variables are 1, 3, 0, 1.

The next theorem, while intuitively evident, is extremely useful for ruling solutions out of consideration when the x_i are bounded.

<u>Theorem 5</u>.[1] Let p denote the greatest integer such that $\sum_{i < p} a_i \alpha_i \leq b$. Then an optimal solution x^* to (1) when the x_i are not required to be integer is given by

$$x_i^* = \begin{cases} \alpha_i & \text{for } i < p \\ 0 & \text{for } i > p \end{cases}$$

$$x_p^* = (b - \sum_{i < p} a_i \alpha_i)/a_p$$

An immediate consequence of Theorem 5 is that it provides a sharper test than the condition $\ddot{M} > M$ when the x_i are bounded. But this also true when the x_i are unbounded, since in problem (2) $x_i \leq [(b - ax^{\circ})/a_i]$ for $i > s$. By Theorem 5, an optimal fractional

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1. The special case of this theorem in which $\alpha_i = 1$ for all i is due to Dantzig [2].

solution x^* to problem (2) when the x_i are unbounded is just

$$x_i^* = \begin{cases} 0 \text{ for } i > s,\ i \neq t,\ i \neq u \\ [(b - ax')/a_i] \quad \text{for } i = t \\ (b - ax' - a_t x_t^*)/a_i \quad \text{for } i = u \end{cases}$$

where t and u are the two smallest i, $i > s$, for which $a_i \leq b - ax'$ $(t < u)$. More generally, Theorem 5 may be applied in this fashion when the variables are bounded by substituting $\beta_i = \min([(b - ax')/a_i], \alpha_i)$ for α_i, restricting i to $i > s$, and replacing b by $b - ax'$. If we denote the optimal objective function value for (2) obtained by Theorem 5 under these stipulations by M^*, then the condition $M^* > M$ is in general stronger than $\bar{M} > M$, and must be satisfied if there exists any feasible (integer) solution $\bar{x}$ such that $c\bar{x} > M$ and $\bar{x}_i = x_i'$ for $i \leq s$.

However, the test $M^* > M$ has a limitation not encountered by the test $\bar{M} > M$. The best one can do in the enumeration process when $M^* \leq M$ is to reduce the value of $\bar{x}_s$ by 1--and then retest to see if it can be reduced further--before defining $\hat{x}$ in terms of $\bar{x}$. This contrasts with the ability to immediately set $\bar{x}_s$ equal to 0 when $\bar{M} \leq M$.

On the other hand, if Theorem 5 is applied relative only to the bounds α_i, then the objective function value so obtained (call it $\bar{\bar{M}}$) yields a test which enables $\bar{x}_s$ to be handled as with the test $\bar{M} > M$. The difference between the power of $\bar{M} > M$ and $\bar{\bar{M}} > M$ should be readily apparent when the α_i are limiting.

It should further be noted that the test $\bar{\bar{M}} > M$ prescribes a significant shortcut in the enumeration process when $x_s' = \alpha_s$. To show this it must first be observed that no test is applied when $x_s' = \alpha_s$ in the Gilmore-Gomory algorithm, since (in addition to the fact that the authors do not consider bounded variables) x' is always defined so that $x_s' = \bar{x}_s - 1$ before testing $\bar{M} > M$. We will outline a method later in which this limitation is absent. For the moment, however, assume that a step exists at which x' is defined to equal $\bar{x}$, and the next vector to be found is constrained to agree with x' in its first s components. The test $\bar{M} > M$ is thus applicable, and if it is failed, all $\bar{x}_i$ for which $w < i \leq s$ may be set equal to 0, where $w = \max(i \mid i \leq s - 1,\ x'_{i+1} \neq a_{i+1})$. The reason for this is that the test $\bar{M} > M$ must continue to be failed as long as no x_i' is increased in value for $i \leq s$. The upper bounds on the x_i prevent such an increase in the x_i' until $\bar{x}$ and x' are redefined and a new value of s is determined satisfying $s \leq w$. This is accomplished for either definition of x' above by setting $\bar{x}_i = 0$ for $w < i \leq s$. The foregoing rule in fact applies when $x_s' \neq a_s$. For then $w = s - 1$, and $\bar{x}_s$ is set equal to 0 in accordance with the observation that $\bar{x}_s$, and hence x_s', cannot increase in the lexicographically decreasing solution sequence until some $\bar{x}_i$ is decreased for $i < s$.

When the problem is structured so that continuous solutions and discrete solutions yield much different values for the objective function, it is sometimes possible to get a sharper test than provided by either $\bar{M} > M$ or $M^* > M$ simply by keeping track of information generated in the enumeration. The key is to take advantage of the fact that the value of s upon which problem (2) is based will be duplicated a number of times. If at some point in the enumeration $b - ax'$ is no larger than at an earlier point when (2) was defined relative to the same s, then any bound on $\sum_{i>s} c_i x_i$ obtained for the earlier problem is an upper bound on this expression for the present problem. The most restrictive permissible bound is of course given either by the best feasible solution found for (2) or by the largest value of M^* for which the test $M^* > M$ was failed.

In spite of the utility of the foregoing results, there are many problems which require the application of additional relations if they are to be solved at all efficiently. In fact, one can readily find problems for which the procedures presented so far can scarcely improve upon complete enumeration. For a very simple example, consider the following.

Example.

Maximize $\quad 80x_1 + 70x_2 + 81x_3 + 60x_4 + 55x_5$

subject to $\quad 43x_1 + 42x_2 + 50x_3 + 41x_4 + 39x_5 \leq 115,$

where the x_i are unbounded.

It can readily be shown that for this problem the Gilmore-Gomory algorithm will have to enumerate all but two of the entire range of feasible solutions. The test $M^* > M$ rules out a few of these solutions but still requires examination of an excessive number of alternatives.[1] To remedy this situation, it is useful to introduce the sets $S_0, S_1, S_2, \dots,$ where we define $S_k = \{ i \mid [b/a_i] = k \}$. Then the following theorem and its corollary allow the foregoing problem to be solved by enumerating only a single solution, and provide a principle which can be used to markedly reduce the number of solutions enumerated in more complex problems.

Theorem 6. Assume that $i \in S_k$ for all i, and let $\beta_i = \max(\alpha_i, [b/a_i])$. Define a subscripted indexing so that $c_{i_1} \geq c_{i_2} \geq \dots \geq c_{i_n}$, and let p be the largest index for which $\sum_{j < p} \beta_{i_j} \leq H$, where $H = \min(k, \sum_{i \leq n} \beta_i)$.

Then the optimal solution x^* to (1) is given by

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1. Fortunately, the example is simple enough that not many feasible alternatives exist. However, far worse examples can easily be constructed.

$$x_{i_j}^* = \begin{cases} \beta_{i_j} & \text{for } j < p \\ 0 & \text{for } j > p \end{cases}$$

$$x_{i_p}^* = N - \sum_{j < p} \beta_{i_j}$$

Corollary to Theorem 6. Assume that the x_i are unbounded, and that $i \in S_k$ for all i. Further let $c_q = \max(c_i)$. Then the optimal solution x^* to (1) is

$$x_i^* = \begin{cases} k & \text{for } i = q \\ 0 & \text{for } i \neq q. \end{cases}$$

From the corollary it can be immediately seen that the optimal solution to the foregoing problem is obtained by setting $x_3 = 2$. In most problems one would not expect the condition $i \in S_k$ to be met for all i. However, advantage can be taken of the preceding theorem by noting that the sets S_0, S_1, ... form a natural partition of the i for $i \leq n$. Clearly we can find the optimal solution to (1) under the restriction that $x_i > 0$ only if $i \in S_k$. Having done so, it follows that if a better solution to (1) exists, then $x_i > 0$ for at least one $i \notin S_k$. Thus, in the enumeration process all solutions may be ignored which involve $x_i > 0$ only for $i \in S_k$. This of course applies at each stage of the enumeration, for Theorem 6 may be as well stated in terms of problem (2): that is, if $S_k = \{ i \mid [(b - ax')/a_i] = k \}$, $\beta_i = \min(\alpha_i, [(b - ax')/a_i])$, and if all indices i are restricted to $i > s$, then the solution specified by the theorem is optimal for (1) subject to the restriction that $x_i' = x_i'$ for $i \leq s$.

To illustrate specifically how this information may be exploited, suppose that the enumeration process is based upon a lexicographically decreasing sequence of solutions as with the Gilmore-Gomory method. Let x' be defined in terms of $\bar{x}$ as earlier (i.e., $x_s' = \bar{x}_s - 1$ and x' and $\bar{x}$ agree in all other components), and let $r = \min(i \mid i > s \text{ and } a_i \leq b - ax')$. Suppose $r \in S_k$, where S_k is defined relative to problem (2) (replacing b by $b - ax'$, etc., in Theorem 6). Under the assumption that the test $M^* > M$ is passed, apply Theorem 6 to find the optimal solution to (2) under the restriction that $x_i > 0$ only if $i \in S_k$. Then to find a better solution we next seek the lexicographically largest x such that $x_i = x_i'$ for $i \leq s$ and $x_i > 0$ for at least one i such that $i \notin S_k$. This solution, which we denote by $\bar{\bar{x}}$, is given as follows.

Let $L = \min(a_i \mid i > r, i \notin S_k)$. If L is not well-defined, $\bar{\bar{x}}$ does not exist and the optimal solution to (2) is given by Theorem 6. Otherwise,

$$\bar{\bar{x}}_i = \begin{cases} x_i' \text{ for } i \leq s \\ \min(a_i, [(b - \sum_{j=1}^{i-1} a_j \bar{\bar{x}}_j - L)/a_i]) \quad \text{for } s+1 \leq i < q \\ \min(a_i, [(b - \sum_{j=1}^{i-1} a_j \bar{\bar{x}}_j)/a_i]) \quad \text{for } i \geq q \end{cases}$$

where $q = \min(i \mid i > r, i \notin S_k, \text{ and } a_i \leq b - \sum_{j=1}^{i-1} a_j \bar{\bar{x}}_j)$, i.e., q is "discovered" in the process of assigning values to the $\bar{\bar{x}}_i$.

By redefining $\bar{x}$ to be equal to $\bar{\bar{x}}$ instead of equal to $\hat{x}$ when the tests are passed and the optimal solution is found on S_k, segments of the solution space which might otherwise be enumerated can thus be bypassed. In the previous example problem, if $a_5 = 39$ is replaced by $a_5 = 38$, the corollary to Theorem 6 no longer applies and we still have a situation in which the Gilmore-Gomory algorithm enumerates nearly every feasible solution. However, by using the enumeration procedure in which

x is followed by $\bar{\bar{x}}$, only 6 solutions need to be examined.

Theorem 6 will of course have less application in those problems for which M is more often equal to $\sum_{i \leq n} \beta_i$ than to k, a situation that may occur if the α_i are small and the a_i cover a wide range of sizes. For the problems in which Theorem 6 does prove useful, however, it may be preferable not to follow $\ddot{x}$ with $\bar{\bar{x}}$ immediately, but to generate only the first nonzero component of $\bar{\bar{x}}$ for $i > s$ (call it $\bar{\bar{x}}_h$), and redefine $\bar{x}$ and x' so that

$$x_i' = \begin{cases} \ddot{x}_i & \text{for } i \leq h \\ 0 & \text{for } i > h, \end{cases} \qquad \text{and} \qquad \bar{x} = x'.$$

The tests $M^* > M$ and $\bar{\bar{M}} > M$ may then be reapplied, and if they are passed, the new $\bar{\bar{x}}$ to follow the current $\bar{x}$ determined in its first h components.[1]

In addition to giving the tests more extensive application, an advantage of this procedure is the ability it affords to skip examination of those solutions which lie between $\ddot{x}$ and $\bar{\bar{x}}$ at each level.

To give a clearer idea of how the foregoing procedure might be incorporated into an enumerative algorithm, we diagram such a procedure below.[2]

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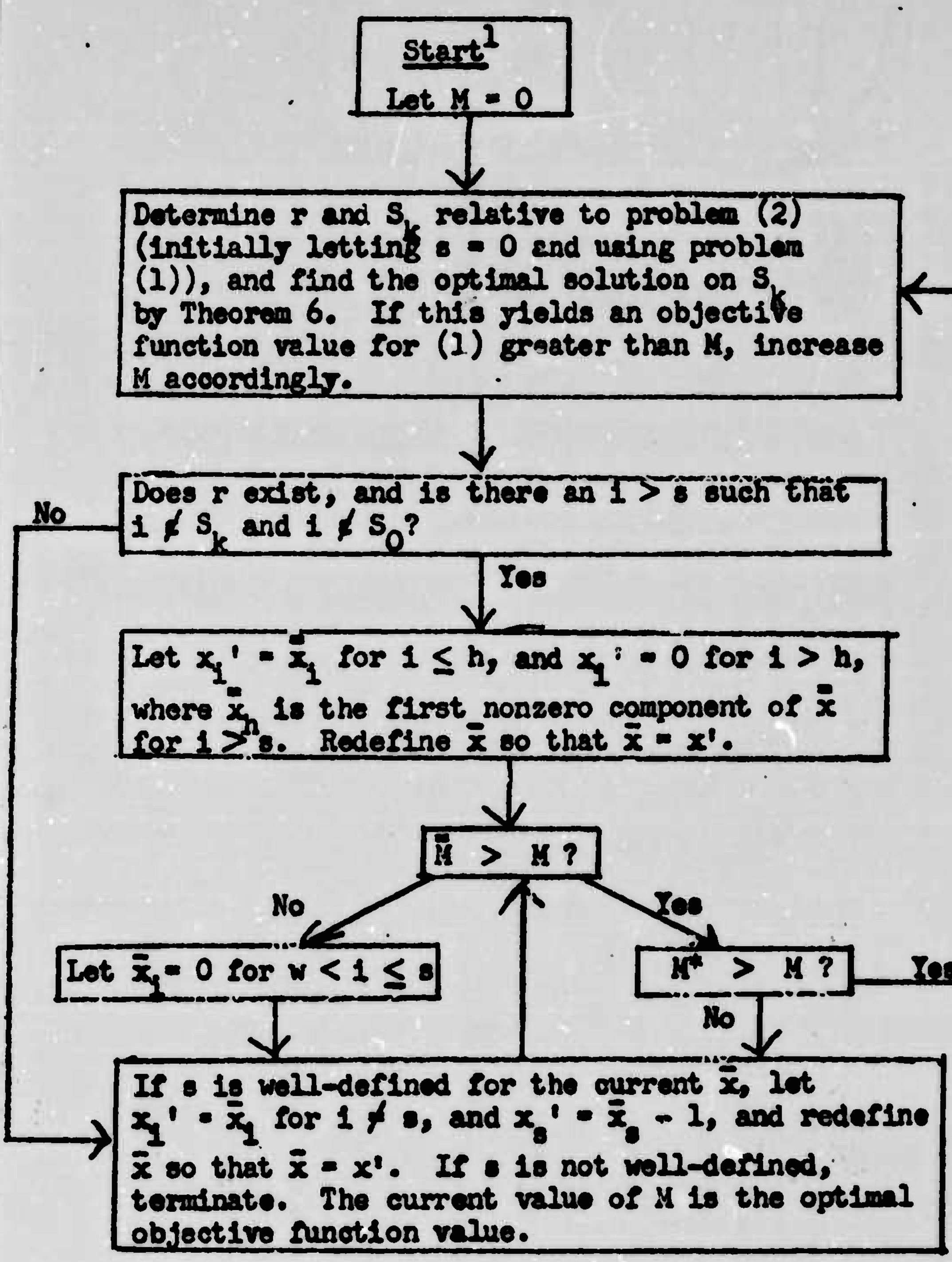
1. Likewise, in the Gilmore-Gomory algorithm it would sometimes seem to be preferable to reapply the test $\bar{M} > M$ as successive nonzero components of $\hat{x}$ are generated. A simple decision rule for this situation would be to establish a number δ ($\delta > 0$) and reapply the test $\bar{M} > M$ for successive components of $\hat{x}$ only if $\bar{M} < \delta + M$. Similar remarks apply to testing $\bar{\bar{M}} > M$ and $M^* > M$. It should be noted, however, that replacing $\bar{x}$ by x may significantly change the test situation, so that the decision rule in this case might need to be more complex to be effective.

2. Without the inclusion of additional rules, this procedure can be applied only with difficulty to the simultaneous solution of several knapsack problems in the manner outlined by Gilmore and Gomory. The reason for this is that the solution x defined relative to $b = b^*$, unlike the solution $\hat{x}$, is not always lexicographically greater than or equal to the solution $\bar{\bar{x}}$ defined relative to $b < b^*$.



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1. The definitions of s, r and S_k are those used in defining $\bar{\bar{x}}$ above, and the definition of w is given in the dicussion of $\bar{\bar{M}}$. The tests $\bar{\bar{M}} > M$ and $M^* > M$ are applied to x' as it is specifically defined in the algorithm.

The purpose of the foregoing diagram is of course solely illustrative. It is not to be construed as a representation of a highly efficient algorithm, since it makes no use of many of the results of this paper, and fails to employ rules for avoiding computational duplications or for bypassing the tests when they are u... 'ely to yield results.

In addition to the procedure outlined in the diagram, other more refined results may also be derived from Theorem 6. For example, suppose that the test $\bar{\bar{M}} > M$ is applied under the provisional assumption that $x_r = k$ ($r \in S_k$) and $x_i = 0$ for $i \in S_k$, $i \neq r$, where r is defined as above. If the test is failed, it follows that for any optimal solution x^* to (2), $x_r^* < k$ or $x_i^* = 0$ for all $i \notin S_k$. For suppose that $x_r^* = k$. Then in order to satisfy $ax \leq b$, $x_i^* = 0$ for all $i \in S_k$ except $i = r$. But $x_i^* > 0$ for $i \notin S_k$ is then impossib le, since the test $\bar{\bar{M}} > M$ was conducted under precisely these assumptions. Moreover, the failure of the test implies that if i replaces r in the test for any $i \in S_k$, $i > r$, the test will continue to be failed as long as $a_i \leq a_r$. More generally, there can be no assignment of values to those x_i for $i \in S_k$ and $i \geq r$ which yields $\Sigma\, a_i x_i \leq ka_r$, where the summation is over $i \in S_k$, $i > r$. The fact that $x_i > 0$ is required for at least one $i \notin S_k$ still holds (provided the optimum on S_k has been found), and may be used to develop still further restrictions.

<u>Conclusion</u>.

The objective of this paper has been to develop relations which may be used to improve the efficiency of algorithms for the knapsack problem. Results derived have been illustrated principally in terms of a lexicographically decreasing solution sequence such as employed by Gilmore and Gomory, although we have shown that it can be advantageous to depart from the way in which Gilmore and Gomory define consecutive solutions: e. g., employing Theorem 6 and following $\bar{x}$ with $\ddot{x}$ instead of $\hat{x}$ makes it possible to solve certain problems after examining only a fraction of the alternatives enumerated by Gilmore and Gomory. Although our results may be used in a variety of ways, the actual design of a specific algorithm--except for purposes of illustration--has not been undertaken.[1]

1. For a specific algorithm which deals with the integer programming problem more generally, and which is based in part on results found in this paper, see [5].

APPENDIX

<u>Proof of Theorem 1</u>. Suppose that $x_k^* = p > h_k$ for some optimal solution x^* to (1). Then $x_i^* \le [(b - pa_k)/a_i]$ for each $i \ne k$. Define $q = [p/h_k]$, and let $\bar{x}_k = x_k^* - qh_k$, and $\bar{x}_i = x_i^* + qh_i$ for $i \ne k$. Thus $0 \le \bar{x}_k < h_k$, and clearly $c\bar{x} \ge cx^*$ and $a\bar{x} \le ax^*$ by (ii) and (iii), hence $\bar{x}$ is optimal provided $\bar{x}_i \le \alpha_i$ for $i \ne k$. This is assured provided
$[(b - pa_k)/a_i] + qh_i \le \alpha_i$ for $h_i \ne 0$ and $i \ne k$. When $p = 1$, $p \ge h_k > 0$ implies $q = 1$ and the above relation holds by (iv). In general, we note that $a_k/a_i \ge h_i/h_k$ by (iii), and since $p/h_k \ge [p/h_k] = q$, it follows that $(p - h_k)a_k/a_i \ge (p - h_k)h_i/h_k \ge h_i q - h_i$. Adding $b/a_i + h_i - pa_k/a_i$ to the first and last expressions yields
$(b - h_k a_k)/a_i + h_i \ge (b - pa_k)/a_i + qh_i$. But then
$[(b - h_k a_k)/a_i + h_i] \ge [(b - pa_k)/a_i + qh_i]$, and since h_i and qh_i are integer, $[(b - h_k a_k)/a_i] + h_i \ge [(b - pa_k)/a_i] + qh_i$, hence $\bar{x}_i \le \alpha_i$ by (iv). When $\sum_{i \ne k} h_i c_i > h_k c_k$, we obtain $c\bar{x} > cx^*$, proving the last part of the theorem by contradiction.

<u>Proof of Theorem 2</u>. Assume that for every optimal solution x^*, $x_i^* \ge \bar{x}_i$ for all $i \notin P_q$, and $x_q^* > \bar{x}_q$. Then for every such solution, $x_i^* < \bar{x}_i$ for at least one $i \in P_q$ (say for $i = p$) in order to satisfy $ax^* \le b$. Let x^* specifically denote an optimal solution for which x_q^* assumes its smallest value. Then define x' by $x_i' = x_i^*$ if $i \ne p$ and $i \ne q$, $x_q' = x_q^* - 1$, and $x_p' = x_p^* + \langle c_q/c_p \rangle$. Clearly $ax' \le ax^*$, $cx' \ge cx^*$, and $x_p' \le \bar{x}_p + \langle c_q/c_p \rangle - 1 \le \alpha_p$, contrary to the assumption that $x_p \ge x_q^*$ in any optimal solution. When $[c_q/c_i] + 1$ replaces $\langle c_q/c_i \rangle$, the second part of the theorem immediately follows by letting x^* denote any optimal solution in which (i) and (ii) are both false, and noting that this implies the contradiction $cx' > cx^*$.

Proof of Theorem 3. Suppose that for every optimal solution x^*, $x_p^* < \bar{x}_p$ and $x_i^* \leq \bar{x}_i$ for all $i \notin Q_p$. Let it further be assumed that x^* denotes an optimal solution in which x_p^* assumes its largest value. Since $cx^* > c\bar{x}$ (or else $x_p^* = \bar{x}_p$ would be possible), $x_i^* > \bar{x}_i$ for some $i \in Q_p$, say for $i = q$. But then in the solution x' defined by $x_i' = x_i^*$ for $i \neq p$ and $i \neq q$, $x_q' = x_q^* - 1$, and $x_p' = x_p^* + \langle c_q/c_p \rangle$, we have $cx' \geq cx^*$, $ax' \leq ax^*$, and $x_p' \leq \alpha_p$, contrary to the assumption that $x_p \leq x_p^*$ in any optimal solution to (1). When $\langle c_q/c_p \rangle$ is replaced by $[c_q/c_p] + 1$, the second part of the theorem follows analogously, letting x^* denote any optimal solution in which (i) and (ii) are both false, and noting that $cx' > cx^*$.

Proof of Corollary to Theorem 3. Note that for $p \leq s$ and $i > s$ $\alpha_p - \bar{x}_p + 1 = 1 = \langle c_i/c_p \rangle$, and hence $\langle c_i/c_p \rangle a_p \leq a_i$. Thus for each $p \leq s$, Q_p includes all $i > s$, and there exists an optimal solution x^* in which (i) $x_p^* \geq \bar{x}_p$ or (ii) $x_i^* > \bar{x}_i$ for some $i \notin Q_p$, hence for some $i \leq s$. Since (ii) is impossible, for each $p \leq s$ there exists an optimal solution in which $x_p^* = \bar{x}_p$. Select one such p, say $p = q$, and replace the variable x_q by the constant $\bar{x}_q$, reducing the original problem to a new one. Clearly the optimal solution to this latter problem in conjunction with $x_q = \bar{x}_q$ yields an optimal solution to the original problem. But for the new problem, Theorem 3 again implies that for each remaining p such that $p \leq s$, $x_p^* = \bar{x}_p$ in some optimal solution x^*. Thus we may repeat the process of selecting one such p, denoted by q, and assigning the variable x_q the constant value $\bar{x}_q$, continuing until no more p remain which satisfy $p \leq q$. But since $a_i > b - a\bar{x}$ for $i > s$, there is no solution x' in which $x_p' = \bar{x}_p$ for $p \leq s$ and $x_i' > 0$ for $i > s$. Hence $\bar{x}$ is optimal.

Proof of Theorem 4. Suppose that for every optimal solution x^*, $x_q^* \geq h$ and $x_i^* \leq \bar{x}_i$ for all $i \leq s$. Consider the problem: maximize cx, subject to $ax \leq a\bar{x}$ $(= b - G)$, and $x_i \leq \bar{x}_i$ for $i \leq s$. In this latter problem assume also that a_q is replaced by $a_q - G/h$. Since $x_q^* \geq h$ in the original problem, then the same solution x^* yields $ax^* \leq b - G$ in the new problem. But $\bar{x}$ is optimal for the new problem by Theorem 5 (proved below). Hence $c\bar{x} \geq cx^*$, and $\bar{x}$ is in fact optimal for the original problem, contrary to the fact that $\bar{x}_q = 0 < h$. The last part of the theorem follows similarly, letting x^* denote a single solution for which the theorem is supposedly false, and concluding $c\bar{x} > cx^*$.

Proof of Theorem 5. Assume the theorem false, and let $\bar{x}$ denote a feasible solution to (1) for which $c\bar{x} > cx^*$. Let Σ' denote the summation over those i for which $\bar{x}_i > x_i^*$, and let Σ'' denote the summation over those i for which $\bar{x}_i < x_i^*$. Then $c\bar{x} - cx^* = \Sigma' c_i(\bar{x}_i - x_i^*) + \Sigma'' c_i(\bar{x}_i - x_i^*)$

$= \Sigma' a_i(\bar{x}_i - x_i^*)c_i/a_i + \Sigma'' a_i(\bar{x}_i - x_i^*)c_i/a_i$

$\leq (c_p/a_p) \Sigma' a_i(\bar{x}_i - x_i^*) + (c_p/a_p) \Sigma'' a_i(\bar{x}_i - x_i^*)$

$= (c_p/a_p)(a\bar{x} - ax^*)$, since from the definition of x^* it follows that all i associated with Σ' satisfy $i \leq p$, and all i associated with Σ'' satisfy $i \geq p$. But $ax^* = \sum_{i<p} a_i\alpha_i + a_p(b - \sum_{i<p} a_i\alpha_i)/a_p = b$, and hence $a\bar{x} - ax^* \leq 0$. Thus $c\bar{x} - cx^* \leq 0$, contrary to assumption.

Proof of Theorem 6. We note that $\sum_{i \leq n} x_i^* = H$, $x_i^* \leq \beta_i$ for all i, and that $\sum_{i \leq n} a_i x_i^* \leq a_q H \leq a_q k \leq b$, where $a_q = \max(a_i)$. Due to the definition of the subscripted indexing, $\sum_{i \leq n} c_i \bar{x}_i \leq \sum_{i \leq n} c_i x_i^*$ for any solution $\bar{x}$ such that $\sum_{i \leq n} \bar{x}_i \leq H$. Clearly $\sum_{i \leq n} \bar{x}_i \leq \sum_{i \leq n} \beta_i$ if $\bar{x}$ is feasible. Hence it remains to show only that $\sum_{i \leq n} \bar{x}_i \leq k$ for any feasible solution $\bar{x}$. Suppose on the contrary that $\sum_{i \leq n} \bar{x}_i \geq k + 1$, and let $a_p = \min(a_i)$. Then $\sum_{i \leq n} a_i \bar{x}_i \geq a_p \sum_{i \leq n} \bar{x}_i \geq a_p(k + 1) > b$, which is impossible.

<u>Proof of the Corollary to Theorem 6.</u> The corollary is a special case of the theorem.

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